LINEAR INTEGRAL OF A CYCLIC DISPLACEMENT FOR A GYROSCOPE ON GIMBALS

(LINEINYI INTEGRAL TSIKLICHESKOGO PEREMESHCHENIIA DLIA GIROSKOPA V KARDANOVOM PODVESE)

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The cyclic displacements introduced in mechanics by Chetaev [1] give the first integrals of the equations of motion. These displacements are determined from the properties of the introduced infinitesimal operators of the group of infinitely small Lie transformations, from the kinetic energy and from the force function.

1. Let us consider a symmetric gyroscope on gimbals [2 and 3]. The center of gravity of the gyroscope with its casing (the inner ring) is on the axis of symmetry of the gyroscope (Fig.1).

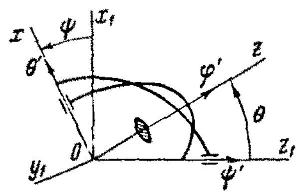

Fig. 1

The z_1-axis of the fixed coordinate system $Ox_1y_1z_1$ is along the gimbals' outer ring axis, the origin coincides with the fixed point of the gyroscope O. The moving coordinate system $Oxyz$, with its origin coiciding with the fixed point of the gyroscope, is connected with the axes of the casing. The x-axis is along the casing's axis of rotation, the z-axis is along the gyroscope's axis of symmetry.

Let the x, y, z axes be the principal axes of the casing's ellipsoid of inertia with respect to the fixed point O.

Let us introduce the following notation: ψ is the angle of rotation of the outer gimbal ring, θ is the angle of rotation of the casing in the ring, φ is the angle of rotation of the gyroscope in the casing (angle of spin of the gyroscope in its casing), A^0, B^0, C^0 are the principal moments of inertia of the casing, A, $B = A$, C are the moments of inertia of the gyroscope about the x, y, z axes, finally I is the moment of inertia of the outer gimbal ring about the z_1-axis. We shall assume that the ellipsoid of inertia of the gyroscope about the point O is an ellipsoid of revolution about the z-axis.

The x, y, z components of the instanteneous angular velocity of rotation of the casing are

$$p^\circ = \theta', \qquad q^\circ = \psi' \sin\theta, \qquad r^\circ = \psi' \cos\theta$$

and the x, y, z components of the instanteneous angular velocity of the gyroscope are

$$p = \theta', \qquad q = \psi' \sin\theta, \qquad r = \varphi' + \psi' \cos\theta$$

The kinetic energies of the outer gimbal ring, of the casing and the gyroscope, multiplied by 2 are, respectively

$$I\psi'^2, \qquad A^\circ p^{\circ 2} + B^\circ q^{\circ 2} + C^\circ r^{\circ 2} \qquad Ap^2 + Bq^2 + Cr^2$$

The total kinetic energy of the system, multiplied by 2, is

$$2T = (A + A^\circ)\,\theta'^2 + [I + C^\circ + (A + B^\circ - C^\circ)\sin^2\theta]\,\psi'^2 + C\,(\varphi' + \psi'\cos\theta)^2$$

Assuming absence of friction in the bearings, the active forces acting on the system admit the force function U, depending in general on the angles θ, ψ and φ.

2. We shall apply a method which has been previously used for finding the linear integral of a rigid body containing inside moving masses [4]. We shall introduce the infinitesimal operators of the Lie group of infinitely small transformations in the following way.

Let the state of our mechanical system be determined by the parameters of Rodriguez [5] which, for our system, will be regarded as Poincaré's variables [1 and 6],

$$\lambda = \tau \sin {}^1/_2\theta \cos {}^1/_2(\psi - \varphi), \qquad \mu = \tau \sin {}^1/_2\theta \sin {}^1/_2(\psi - \varphi)$$
$$\nu = \tau \cos {}^1/_2\theta \sin {}^1/_2(\psi + \varphi), \qquad \rho = -\tau \cos {}^1/_2\theta \cos {}^1/_2(\psi + \varphi)$$

These variables are connected by the relation

$$\lambda^2 + \mu^2 + \nu^2 + \rho^2 = \tau^2 \qquad (\tau = \text{const})$$

Without losing generality we can set for simplicity $\tau = 1$. The parameters of the real displacements will be

$$\eta_1 = \frac{1}{4}\left(\frac{\psi'}{a} - \frac{\varphi'}{b}\right), \qquad \eta_2 = \frac{1}{4}\left(\frac{\psi'}{a} + \frac{\varphi'}{b}\right), \qquad \eta_3 = \frac{\theta'}{2}$$
$$(a = \text{const} \neq 0, \quad b = \text{const} \neq 0) \tag{1}$$

The Poincaré variables satisfy relations

$$d\lambda\,/\,dt = -\mu\,[(a + b)\,\eta_1 + (a - b)\,\eta_2] + \lambda\varkappa^{-1}\eta_3$$
$$d\mu\,/\,dt = \lambda\,[(a + b)\,\eta_1 + (a - b)\,\eta_2] + \mu\varkappa^{-1}\eta_3 \qquad \varkappa = \left(\frac{\lambda^2 + \mu^2}{\nu^2 + \rho^2}\right)^{1/2}$$
$$d\nu\,/\,dt = -\rho\,[(a - b)\,\eta_1 + (a + b)\,\eta_2] - \nu\varkappa\eta_3$$
$$d\rho\,/\,dt = \nu\,[(a - b)\,\eta_1 + (a + b)\,\eta_2] - \rho\varkappa\eta_3$$

The differential of the function of the state of the mechanical system $f(t, \lambda, \mu, \nu, \rho)$ with respect to the real displacements of the system is determined from

$$df = \left(X_0 + \sum_{i=1}^{3} \eta_s X_s f\right) dt$$

where the infinitesimal operators of the Lie group of the real displacements are

$$X_0 = \frac{\partial}{\partial t}$$
$$X_1 = (a + b)\left(\lambda\frac{\partial}{\partial\mu} - \mu\frac{\partial}{\partial\lambda}\right) + (a - b)\left(\nu\frac{\partial}{\partial\rho} - \rho\frac{\partial}{\partial\nu}\right)$$
$$X_2 = (a - b)\left(\lambda\frac{\partial}{\partial\mu} - \mu\frac{\partial}{\partial\lambda}\right) + (a + b)\left(\nu\frac{\partial}{\partial\rho} - \rho\frac{\partial}{\partial\nu}\right) \tag{2}$$
$$X_3 = \varkappa^{-1}\left(\lambda\frac{\partial}{\partial\lambda} + \mu\frac{\partial}{\partial\mu}\right) - \varkappa\left(\nu\frac{\partial}{\partial\nu} + \rho\frac{\partial}{\partial\rho}\right)$$

The operators of the group of the real displacements

$$X_0,\ X_1,\ X_2,\ X_3$$

are commutative, likewise the operators of the subgroup of the possible displacements

$$X_1,\ X_2,\ X_3$$

that is

$$(X_i X_k) = 0 \qquad (X_i,\ X_0) = 0 \qquad (i,\ k = 1,\ 2,\ 3) \tag{3}$$

3. We shall apply the constructed infinitesimal operators to find the integral of the cyclic displacement in the problem stated in Section 1.

From Formulas (1) follows

$$\psi' = 2a\,(\eta_1 + \eta_2), \qquad \varphi' = -\,2b\,(\eta_1 - \eta_2), \qquad \theta' = 2\eta_3$$

The kinetic energy of our gyroscope on gimbals in the Poincaré-Chetaev variables can be written as

$$\begin{aligned} T = 2\,\{a^2K + [a\,(\nu^2 + \rho^2 - \lambda^2 - \mu^2) - b]^2\,C\}\,\eta_1^2 + \\ + 2\,\{a^2K + [a\,(\nu^2 + \rho^2 - \lambda^2 - \mu^2) + b]^2\,C\}\,\eta_2^2 + 2\,(A + A^\circ)\,\eta_3^2 + \\ + 4\,\{a^2K + [a^2\,(\nu^2 + \rho^2 - \lambda^2 - \mu^2)^2 - b^2]\,C\}\,\eta_1\eta_2 \end{aligned} \tag{4}$$

Here

$$K = I + C^\circ + 4\,(A + B^\circ - C^\circ)\,(\lambda^2 + \mu^2)\,(\nu^2 + \rho^2)$$

The displacement X_1 from (2) will according to Chetaev be cyclic, as shown by (3), which is always satisfied, and by requiring that $X_1(L) = 0$, which is satisfied if we set

$$X_1(U) = 0$$

Similar requirement in the Euler variables gives

$$a\frac{\partial U}{\partial \psi} - b\frac{\partial U}{\partial \varphi} = 0 \tag{5}$$

Under condition (5), the cyclic displacement X_1 has the following integral

$$\begin{aligned} \frac{1}{4}\frac{\partial T}{\partial \eta_1} = \{a^2K + [a\,(\nu^2 + \rho^2 - \lambda^2 - \mu^2) - b]^2\,C\}\,\eta_1 + \\ + \{a^2K + [a^2\,(\nu^2 + \rho^2 - \lambda^2 - \mu^2)^2 - b^2]\,C\}\,\eta_2 = \mathrm{const} \end{aligned} \tag{6}$$

In Euler's variables the integral has the form

$$a\,\{[I + C^\circ + (A + B^\circ - C^\circ)\sin^2\theta]\,\psi' + Cr\cos\theta\} - b\,Cr = \mathrm{const} \tag{7}$$

This integral has not been discussed in the literature.

Thus, requiring X_2 from (2) to be a cyclic displacement we can show that under condition

$$a\frac{\partial U}{\partial \psi} + b\frac{\partial U}{\partial \varphi} = 0 \tag{8}$$

we can have the integral

$$a\,\{[I + C^\circ + (A + B^\circ - C^\circ)\sin^2\theta]\,\psi' + Cr\cos\theta\} + bCr = \mathrm{const} \tag{9}$$

Besides these integrals shown above the equations of motion of a gyroscope on gimbals permit the known kinetic energy integral

$$[I + C^\circ + (A + B^\circ - C^\circ)\sin^2\theta]\,\psi'^2 + (A + A^\circ)\,\theta'^2 + C\,(\varphi' + \psi'\cos\theta)^2 = 2U + h$$

The requirement to have the two cyclic displacemements X_1 and X_2 simultaneously leads to two integrals (7) and (9) under the conditions (5) and (8). The sum and the difference of (5) and (8) give condition

$$\frac{\partial U}{\partial \psi} = \frac{\partial U}{\partial \varphi} = 0$$

The integrals (7) and (9) can be broken up into integrals shown by Chetaev, who reduced the integration of the equations of motion to quadratures and analyzed the problem of stability with respect to the angle of nutation [2].

4. Similar results with the same parameters of possible displacements can be also obtained when the Eulerian angles are used for Poincaré variables.

Knowing the integral of the cyclic displacement (7), we can very simply check it, by using the differential equations of motion of the system

$$(A + A^\circ)\,\theta'' - \psi'^2 (A - C + B^\circ - C^\circ) \sin\theta\cos\theta + C\varphi'\psi' \sin\theta = \frac{\partial U}{\partial \theta}$$

$$\frac{d}{dt}\{[I + C^\circ + (A + B^\circ - C^\circ)\sin^2\theta\,]\,\psi' + C\cos\theta\,(\varphi' + \psi'\cos\theta)\} = \frac{\partial U}{\partial \psi}$$

$$\frac{d}{dt}\, C\,(\varphi' + \psi'\cos\theta) = \frac{\partial U}{\partial \varphi}$$

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